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O. DIEKMANN

ON A NONLINEAR INTEGRAL EQUATION ARISING IN MATHEMATICAL EPIDEMIOLOGY

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On a nonlinear integral equation arising in mathematical epidemiology*)

by

O. Diekmann

ABSTRACT

In this note we study a nonlinear integral equation of mixed Volterra-Fredholm type describing the spatio-temporal development of an epidemic. The emphasis is on qualitative aspects.

We prove that the equation exhibits the hair-trigger effect if a certain parameter exceeds a threshold value. Subsequently we pay attention to travelling wave solutions. These are found as solutions of a certain nonlinear convolution equation on the line. We show that there exists c_0, $0 < c_0 < \infty$, such that a travelling wave solution with speed c exists if $|c| > c_0$. Moreover, using Tauberian methods we prove that this solution is unique (modulo translation) and that no travelling wave solutions with speed c do exist if $|c| < c_0$.

In this note most proofs are merely outlined. More general results and complete proofs will be published elsewhere.

KEY WORDS & PHRASES: *spread of infection in space and time; nonlinear integral equation of mixed Volterra-Fredholm type; threshold phenomenon; hair-trigger effect; travelling waves; nonlinear convolution equation; existence and uniqueness (modulo translation) or nonexistence of nontrivial solutions; Tauberian methods.*

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1. INTRODUCTION

In this note we shall study some qualitative aspects of the development of an epidemic in space and time. The mathematical problems that we shall come across are mainly those of proving existence or nonexistence of solutions of nonlinear convolution equations. We shall give a survey of some of the results that we obtained in [2] and, jointly with H.G. Kaper, in [3]; all details that we omit here can be found there.

In the space-independent Kermack and McKendrick model the evolution of the epidemic is governed by the equation

$$u(t) = S_0 \int_0^t g(u(t-\tau))A(\tau)d\tau + \int_0^t h(\tau)d\tau, \qquad 0 \le t < \infty, \tag{1.1}$$

where

$$u(t) = -\ln \frac{S(t)}{S_0}, \text{ S is the density of susceptibles,}$$

$$g(y) = 1 - e^{-y},$$

and A,h are given nonnegative functions describing, respectively, the infectivity of an individual which has been infected at $t = 0$ and the influence of the history up to $t = 0$. Suppose

$$\int_0^\infty A(\tau)d\tau = \gamma < \infty,$$

$$\int_0^\infty h(\tau)d\tau = H(\infty) < \infty,$$

then, under appropriate hypotheses, $u(t) \to u(\infty)$ as $t \to \infty$ and

$$u(\infty) = \gamma S_0 g(u(\infty)) + H(\infty). \tag{1.2}$$

Equation (1.2) has a unique positive solution $u(\infty)$ for each positive $H(\infty)$. Let $\underline{u}$ be defined by

$$\underline{u} = \inf_{H(\infty)>0} u(\infty)$$

then clearly

$$\underline{u} = \gamma S_0 g(\underline{u}). \tag{1.3}$$

The most important qualitative feature of the model is the so-called *treshold phenomenon*

$$\underline{u} > 0 \quad \text{if and only if} \quad \gamma S_0 > 1.$$

(Plotting a picture will make this evident.) The fact that this result is biologically significant appears from

$$\frac{S(\infty)}{S_0} < e^{-\underline{u}},$$

or, in words: the fraction of the susceptible population that escapes from getting the disease is less than $\exp(-\underline{u})$ for any initial infectivity (no matter how small).

Following Kendall we introduce space-dependence in the model by assuming that the infectivity is in fact a weighted spatial average

$$(1.4) \qquad u(t,x) = S_0 \int_0^t A(\tau) \int_{\mathbb{R}^n} g(u(t-\tau,\xi))V(x-\xi)d\xi d\tau + \int_0^t h(\tau,x)d\tau,$$

where $V: \mathbb{R}^n \to \mathbb{R}$ is a nonnegative radial function, and

$$\int_{\mathbb{R}^n} V(x)dx = 1.$$

An obvious question is now: is there an analogous threshold phenomenon for equation (1.4)? We shall show that the answer is yes if $n = 1$ or $n = 2$. In particular we find that, if $\gamma S_0 > 1$, the equation exhibits the *hair-trigger effect*: no matter how little infectivity is introduced in an arbitrarily small subset of $\mathbb{R}^n$, eventually there will be a large effect at every point.

Subsequently we shall investigate the possibility of *travelling wave solutions*. Instead of equation (1.4), describing an initial value problem, we then consider the time-translation invariant equation

$$(1.5) \qquad u(t,x) = S_0 \int_0^\infty A(\tau) \int_{\mathbb{R}^n} g(u(t-\tau,\xi))V(x-\xi)d\xi d\tau, \qquad -\infty < t < \infty.$$

Our main result is that, under appropriate hypotheses, there exists c_0, $0 < c_0 < \infty$, such that (1.5) has a travelling wave solution $u(t,x) = w(x+ct)$ if $|c| > c_0$ and no such solution if $|c| < c_0$.

In the following we shall normalize A and incorporate the constants γ and S_0 in the function g.

2. THE HAIR-TRIGGER EFFECT

Let us consider the equation

$$(2.1) \qquad u(t,x) = \int_0^t A(\tau) \int_{\mathbb{R}^n} g(u(t-\tau,\xi))V(x-\xi)d\xi d\tau + f(t,x),$$

where $u: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is unknown and A, g, V and f satisfy

H_A: $A: \mathbb{R}_+ \to \mathbb{R}$ is nonnegative; $A \in L_1(\mathbb{R}_+)$ and $\int_0^\infty A(\tau)d\tau = 1$.

H_g: $g: \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (uniformly on $\mathbb{R}_+$), monotone nondecreasing and bounded from above; $g(0) = 0$.

H_V: $V: \mathbb{R}^n \to \mathbb{R}$ is nonnegative; $V \in L_1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} V(x)dx = 1$; V is a radial function.

H_f: $f: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is nonnegative and continuous; $f(\cdot,x)$ is monotone nondecreasing for each $x \in \mathbb{R}^n$; $\{f(t,\cdot) \mid t \geq 0\}$ is uniformly bounded and equicontinuous.

Let $BC(\Omega)$ denote the Banach space of the bounded continuous functions on Ω, equipped with the supremum norm.

THEOREM 2.1. *There exists a unique continuous solution* $u: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *of equation* (2.1). *Moreover* u *is nonnegative,* $u(\cdot,x)$ *is monotone nondecreasing for each* $x \in \mathbb{R}^n$ *and there exists* $u(\infty,\cdot) \in BC(\mathbb{R}^n)$ *satisfying*

$$(2.2) \qquad u(\infty,x) = \int_{\mathbb{R}^n} g(u(\infty,\xi))V(x-\xi)d\xi + f(\infty,x),$$

and such that $u(t,x) \to u(\infty,x)$ *as* $t \to \infty$ *uniformly on compact subsets of* $\mathbb{R}^n$.

SKETCH OF THE PROOF. The local (i.e., $t \in [0,T]$, T sufficiently small) existence and uniqueness of a solution follows from a straightforward application of the Banach contraction mapping principle. The nonnegativity and the monotonicity follow from the construction of the solution as the limit of a sequence that is obtained by iteration. The global existence can be established by a continuation procedure and the boundedness by a simple estimate. The boundedness and the monotonicity yield pointwise convergence to a limit as $t \to \infty$ and by application of the Arzela-Ascoli theorem this can be strengthened as stated. Then it is easy to show that the limit has to satisfy equation (2.2). □

The mapping N defined by Nf = u relates the introduced infectivity to the thereby caused effect. Note that $Nf \equiv 0$ if $f \equiv 0$. For each finite T, N is continuous as a mapping from $BC([0,T] \times \mathbb{R}^n)$ into itself. However, as a mapping from $BC(\mathbb{R}_+ \times \mathbb{R}^n)$ into itself, N need not be continuous at $f \equiv 0$. In order to show this we shall investigate the final state equation (2.2). Firstly we state some lemmas which are crucial.

Let $w * k$ denote the convolution of w and k and k^{m*} the (m-1)-times iterated convolution of k with itself. Let $\hat{k}$ denote the Fourier transform of k. Suppose

H_k: $k \in L_1(\mathbb{R}^n)$, k is nonnegative and $\int_{\mathbb{R}^n} k(x)dx = 1$.

Consider the inequality

$$w \geq w * k. \tag{2.3}$$

LEMMA 2.2. *The following statements are equivalent*

(i) *there exists* $w \in BC(\mathbb{R}^n)$ *such that* (2.3) *is satisfied with strict inequality in some point;*

(ii) *there exists* $h > 0$ *such that*

$$\sum_{m=1}^{\infty} \int_{|x|\leq h} k^{m*}(x)dx$$

converges;

(iii) *there exist* $a > 0$, $C < \infty$ *such that*

$$\int_{|x|\leq a} \mathrm{Re}\left(\frac{1}{1+\varepsilon-\hat{k}(x)}\right) dx \leq C$$

for all $\varepsilon > 0$.

It is not easy to check directly whether a given k satisfying H_k has property (iii). Note that $\hat{k}(0) = 1$ and $\hat{k}(x) \neq 1$ for $x \neq 0$. So for $\varepsilon = 0$, the integrand in (iii) has a singularity at the origin and nowhere else. The next lemma establishes conditions on k such that the singularity is integrable.

LEMMA 2.3.

(a) *If* $n = 1$ *and* $\int_{-\infty}^{\infty} |x|k(x)dx < \infty$ *then* (iii) *is equivalent to* $\int_{-\infty}^{\infty} xk(x)dx \neq 0$*;*

(b) *if* $n = 2$ *and* $\int_{\mathbb{R}^2} |x|^2 k(x)dx < \infty$ *then* (iii) *is equivalent to*

$$a\int_{\mathbb{R}^2} x_1 k(x)dx + b\int_{\mathbb{R}^2} x_2 k(x)dx \neq 0$$

for some a *and* b*;*

(c) *if* $n \geq 3$ *then every* k *satisfying* H_k *has property* (iii).

A proof of lemmas 2.2 and 2.3 can be found in Essén [4] and in Feller [5, sections VI.10 and XVIII.7]. The following lemma deals with the case that the inequality (2.3) is in fact an equality.

LEMMA 2.4. *Suppose* $w \in BC(\mathbb{R}^n)$ *satisfies*

(2.4) $$w = w * k,$$

then $w \equiv C$ *for some constant* C.

For a proof see Rudin [6, Theorem 9.13] or Feller [5, section XI.2].

THEOREM 2.5. *Suppose that*

(α) *there exists* $p > 0$ *such that* $g(y) > y$ *for* $0 < y < p$ *and* $g(p) = p$,

(β) $n = 1$ *or* $n = 2$ *and* $\int_{\mathbb{R}^n} |x|^n V(x)dx < \infty$,

(γ) $f(\infty,\cdot)$ *is not identically zero*,

then $u(\infty,x) \geq p$ *for every* $x \in \mathbb{R}^n$.

PROOF. Let w be defined by $w(x) = \min\{u(\infty,x),p\}$, then (2.2) implies that w satisfies (2.3). Since V is a radial function, lemmas 2.2 and 2.3 imply that w has to satisfy (2.4) and hence that $w \equiv C$, $0 \leq C \leq p$. If $0 \leq C < p$ then (2.2) would not be satisfied. Hence $C = p$. □

In the special case that $g(y) = \gamma S_0(1 - \exp(-y))$, the condition ($\alpha$) of Theorem 2.5 is equivalent to $\gamma S_0 > 1$. The lower bound p is independent of the function f. Thus we have demonstrated the hair-trigger effect if $\gamma S_0 > 1$.

3. TRAVELLING WAVES

Throughout the remaining part of the paper we shall assume that

H_n: $n = 1$.

H_g^2: g is two times continuously differentiable and there exists $p > 0$ such that $g(y) > y$ for $0 < y < p$ and $g(p) = p$.

H_V^2: V is continuous and has compact support.

Our results are actually valid under conditions on g and V which are weaker and which may be different for different theorems.

Substitution of $u(t,x) = w(x+ct)$ into equation (1.5) yields, upon some rearranging,

(3.1) $$w(\xi) = \int_{-\infty}^{\infty} g(w(\eta))V_c(\xi-\eta)d\eta, \quad \xi = x + ct,$$

where

(3.2) $$V_c(\xi) = \int_0^{\infty} A(\tau)V(\xi-c\tau)d\tau, \qquad -\infty < \xi < \infty.$$

Since

$$\int_{-\infty}^{\infty} V_c(\xi)d\xi = 1,$$

(3.1) has, for every c, the constant solutions $w \equiv 0$ and $w \equiv p$. By a *non-trivial solution* of (3.1) we shall mean a continuous function w satisfying (3.1), $0 \leq w(x) \leq p$ and neither being identically 0 nor identically p. Theorem 2.5 shows that for $c = 0$ no nontrivial solutions do exist. As c increases, the mass of V_c shifts to the right and in Theorem 3.1 we shall show that a nontrivial solution exists if the distribution of the mass of V_c has become lopsided enough. Because of the symmetry of V we can restrict our attention to positive c.

THEOREM 3.1. *Suppose that* $g(y) \leq g'(0)y$ *for* $0 \leq y \leq p$, *then there exists* c_0, $0 < c_0 < \infty$, *such that for every* $c > c_0$ (3.1) *has a monotone nondecreasing solution* w *satisfying* $w(-\infty) = 0$, $w(\infty) = p$.

SKETCH OF THE PROOF. With the linearized equation

(3.3) $$v(\xi) = g'(0) \int_{-\infty}^{\infty} v(\eta)V_c(\xi-\eta)d\eta,$$

there is associated the characteristic equation

(3.4) $\qquad L_c(\lambda) = 1,$

where

(3.5) $\qquad L_c(\lambda) = g'(0) \int_{-\infty}^{\infty} e^{-\lambda\xi} V_c(\xi)d\xi$

$$= g'(0) \int_0^{\infty} e^{-\lambda c\tau} A(\tau)d\tau \int_{-\infty}^{\infty} e^{-\lambda\xi} V(\xi)d\xi.$$

Note that $L_c(0) = g'(0) > 1$ (since $y < g(y) \le g'(0)y$ for $0 < y \le p$). Real roots of (3.4) yield sign-definite solutions of (3.3). The constant c_0 is defined by

$$c_0 = \inf\{c \mid \text{there exists } \lambda > 0 \text{ such that } L_c(\lambda) = 1\}.$$

The nonnegativity of A and V guarantees that this definition makes sense (for fixed c, $L_c(\lambda)$ is a convex function of λ and for fixed λ it is a monotone decreasing function of c). The basic idea of the proof is to use information obtained from $L_c(\lambda)$ and the properties of g in the construction of two functions ϕ and ψ such that $\phi \le \psi$, $T\phi \ge \phi$, and $T\psi \le \psi$, where T denotes the formal integral operator that is associated with the right-hand side of (3.1). The existence of a solution having the asserted properties is then established by means of an iterative process (T is monotone). □

Similar results have been obtained by Atkinson and Reuter [1] and by Weinberger [7]. Weinberger has also constructed functions ϕ and ψ for the case $c = c_0$.

4. NONEXISTENCE AND UNIQUENESS

The first step towards a proof of the nonexistence of travelling waves with speed less than c_0 is provided by the following lemma concerning the convolution inequality (2.3).

LEMMA 4.1. *Suppose* k *satisfies* H_k *for* n = 1 *and*

$$\int_{-\infty}^{\infty} |x|k(x)dx < \infty, \qquad \int_{-\infty}^{\infty} xk(x)dx \ne 0.$$

Let w *be a bounded and uniformly continuous solution of*

$$w \ge w * k.$$

Then

(i) $\quad w - w * k \in L_1(\mathbb{R})$,

(ii) $\quad \lim_{x\to-\infty} w(x)$ *and* $\lim_{x\to+\infty} w(x)$ *both exist,*

(iii) $\quad w(\infty) - w(-\infty) = \dfrac{\int_{-\infty}^{\infty} (w - w*k)(x)dx}{\int_{-\infty}^{\infty} xk(x)dx}$.

SKETCH OF THE PROOF (see Essén [4] for a detailed proof).
Define

$$n(x) = \begin{cases} \int_x^{\infty} k(\xi)d\xi, & x > 0 \\ -\int_{-\infty}^{x} k(\xi)d\xi, & x \le 0, \end{cases}$$

then $n \in L_1(\mathbb{R})$ and

(4.1) $\qquad w * n(x) - w * n(y) = \int_y^x (w - w*k)(\xi)d\xi.$

From the monotonicity of the right-hand side and the boundedness of the left-hand side of (4.1) there follows (i) and

(4.2) $\qquad w * n(\infty) - w * n(-\infty) = \int_{-\infty}^{\infty} (w - w*k)(\xi)d\xi.$

Since

$$\hat{n}(\lambda) = \frac{1 - \hat{k}(\lambda)}{i\lambda} \text{ for } \lambda \neq 0$$

and

$$\hat{n}(0) = \int_{-\infty}^{\infty} xk(x)dx$$

we know that $\hat{n}(\lambda) \neq 0$ for all λ. Then (ii) and (iii) follow from (4.2) and Pitt's form of Wiener's general Tauberian Theorem (see for example Rudin [6, Theorem 9.7] or Widder [8, Theorem V.10a]. □

Assuming that

H_A^2: $\int_0^\infty \tau A(\tau)d\tau < \infty$

we have

COROLLARY 4.2. *Let* w *be a nontrivial solution of* (3.1) *for some* $c > 0$, *then*

$$\lim_{x\to-\infty} w(x) = 0 \quad \textit{and} \quad \lim_{x\to+\infty} w(x) = p.$$

PROOF. A bounded solution of (3.1) is necessarily uniformly continuous. Since

$$\int_{-\infty}^{\infty} xV_c(x)dx = c \int_0^\infty \tau A(\tau)d\tau > 0,$$

we deduce from Lemma 4.1 that $w(\infty) - w(-\infty) > 0$. From (3.1) and the properties of g it follows that only 0 and p are candidates for being limits. □

THEOREM 4.3. *Let the assumptions of Corollary* 4.2 *be satisfied and suppose* $g'(0) > 1$. *Then there exists* $a > 0$ *such that*

$$\int_{-\infty}^{\infty} w(x)\, e^{-\lambda x}\, dx$$

converges for $0 < \lambda < a$.

SKETCH OF THE PROOF. There exists $\ell > 1$ such that $g(w(x)) > \ell w(x)$ for $x\to-\infty$. Using this inequality and the same kind of arguments as those leading to (i) of Lemma 4.1, one can prove that $w \in L_1((-\infty,0))$ and subsequently by an induction process that

$$\int_{-\infty}^{0} |x|^m\, w(x)dx \le m!\, a^{-m}$$

for some $a > 0$. □

Motivated by Theorem 4.3 we define

$$\bar{\lambda} = \sup\{\lambda \in \mathbb{R} \mid \int_{-\infty}^{\infty} w(x)\, e^{-\lambda x}\, dx \text{ converges}\}$$

and

$$W(\lambda) = \int_{-\infty}^{\infty} w(x)\, e^{-\lambda x}\, dx.$$

The function $W(\lambda)$ is analytic in the strip $0 < \text{Re}\,\lambda < \bar{\lambda}$. As a consequence of the nonnegativity of $w(x)$ we have (see Widder [8, Theorem II.5b])

LEMMA 4.4. *If* $\bar{\lambda} < \infty$, *then* $W(\lambda)$ *is singular in* $\lambda = \bar{\lambda}$.

Writing (3.1) as

$$w(x) = g'(0)\, w * V_c(x) + r(x),$$

where

$$r(x) = \int_{-\infty}^{\infty}\{g(w(\xi)) - g'(0)w(\xi)\}\, V_c(x-\xi)d\xi,$$

we obtain by Laplace transformation

$$W(\lambda) = W(\lambda)L_c(\lambda) + R(\lambda). \tag{4.3}$$

If $\bar{\lambda} < \infty$ then Lemma 4.4 implies that $L_c(\lambda) = 1$ (note that $R(\lambda)$ is regular in a neighbourhood of $\lambda = \bar{\lambda}$). The possibility that $\bar{\lambda} = \infty$ can be excluded by a straightforward but technical proof. Thus we have established the following nonexistence result.

THEOREM 4.5. *Suppose* $c > 0$ *and* $L_c(\lambda) > 1$ *for* $\lambda \geq 0$, *then* (3.1) *has no nontrivial solution.*

Suppose, on the contrary, that the equation $L_c(\lambda) = 1$ has a positive real root, then (4.3) can be used to obtain information concerning the asymptotic behaviour, as $x \to -\infty$, of solutions of (3.1).

THEOREM 4.6. *Suppose* $c > c_0$ *and* $g(y) \leq g'(0)y$ *for* $0 \leq y \leq p$. *Let* σ *denote the smallest positive root of* $L_c(\lambda) = 1$. *Let* w *be a nontrivial monotone nondecreasing solution of* (3.1), *then there exists* $C > 0$ *such that*

$$\lim_{x\to-\infty} w(x)e^{-\sigma x} = C.$$

SKETCH OF THE PROOF. Note firstly that $L_c'(\sigma) \neq 0$ and $R(\sigma) \neq 0$. From (4.3) we obtain

$$W(\lambda) \sim \frac{R(\sigma)}{L_c'(\sigma)(\sigma-\lambda)}, \quad \lambda \uparrow \sigma.$$

By a complex variable Tauberian theorem of the Ikehara type (see for instance Widder [8, Theorem V.17]) we can deduce from this formula the asymptotic behaviour of w(x) as $x \to -\infty$. □

If w(x) is a solution of (3.1), then so is every translate $w_{\bar{x}}(x) = w(x+\bar{x})$ of w. Our final theorem establishes a condition on g such that every monotone nondecreasing nontrivial solution of (3.1) is obtained by translating one specific solution.

THEOREM 4.7. *Suppose* $c > c_0$ and

$$|g(y_1)-g(y_2)| \leq g'(0)|y_1-y_2| \quad \textit{for} \quad 0 \leq y_1,y_2 \leq p.$$

Then there is modulo translation one and only one monotone nondecreasing nontrivial solution of (3.1).

SKETCH OF THE PROOF. Let w_1 and w_2 be two monotone nondecreasing nontrivial solutions. By Theorem 4.6 we can find $\bar{x}$ such that v defined by

$$v(x) = e^{-\sigma x}\{w_1(x) - w_2(x+\bar{x})\}$$

satisfies

$$\lim_{x\to-\infty} v(x) = \lim_{x\to+\infty} v(x) = 0.$$

From

$$|v(x)| \leq g'(0) \int_{-\infty}^{\infty} V_c(\xi)e^{-\sigma\xi}|v(x-\xi)|d\xi$$

one can deduce that |v| cannot assume a maximum. Hence $v \equiv 0$. □

It is an open problem whether *every* nontrivial solution is monotone nondecreasing.

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REMARK. We also draw attention to the recent paper

D.G. ARONSON, *The asymptotic speed of propagation of a simple epidemic*, to appear in: W.E. Fitzgibbon & H.F. Walker, eds., Nonlinear Diffusion (Research Notes in Mathematics, Pitman Publishing Co., 1977).

There it is shown that for a special case of the model c_0 is the asymptotic speed of propagation of infection.